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AFFINE TRANSFORMATIONS OF THE EQUATIONS OF THE LINEAR THEORY OF ELASTICITY

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This paper deals with the general formulas of affine transformations that preserve invariance of the static equations of the linear theory of elasticity in the case of arbitrary anisotropic materials. The invariance of the equations with respect to affine transformations allows one to model a given anisotropic material by another material. All anisotropic materials are divided into classes of mutually congruent materials. The congruency conditions are obtained for orthotropic and isotropic materials and for orthotropic and transversely isotropic materials.

Key words: *affine transformations, anisotropy, elastic moduli, congruency, invariance of equations.*

Some particular cases of using affine transformations to model anisotropic materials by other materials were considered in [1–12]. In the present paper, the general case of affine transformations that preserve invariance of the static equations of the linear theory of elasticity for arbitrary anisotropy is studied.

Three-dimensional static equations of the linear theory of elasticity are written in the Cartesian rectangular coordinates x_1, x_2, and x_3 as

$$\partial_j \sigma_{ij} + F_i = 0, \qquad \sigma_{ij} = A_{ijkl}\varepsilon_{kl}, \qquad \varepsilon_{kl} = (\partial_k u_l + \partial_l u_k)/2, \tag{1}$$

where $\sigma_{ij} = \sigma_{ji}$ is the stress tensor, $\varepsilon_{kl} = \varepsilon_{lk}$ is the strain tensor, $A_{ijkl} = A_{jikl} = A_{klij}$ is the elastic-modulus tensor, u_k is the displacement vector, F_i is the body-force vector, and ∂_k is the derivative with respect to x_k. Summation is performed over the repeated indices. From Eqs. (1), one obtains the equations in displacements

$$A_{i(kl)j}\partial_{kl}u_j + F_i = 0, \tag{2}$$

where $A_{i(kl)j} = (A_{iklj} + A_{ilkj})/2$ and $L_{ij} = A_{i(kl)j}\partial_{kl}$ is the symmetric operator ($L_{ij} = L_{ji}$). The boundary surface

$$f(x_1, x_2, x_3) = 0 \tag{3}$$

is subjected to the boundary conditions

$$\sigma_{ij}n_j = p_i \qquad \text{or} \qquad u_i = u_i^0, \tag{4}$$

where p_i is the vector of external forces applied to the body surface, u_i^0 is the vector of the boundary displacements, and n_i is the unit outward normal vector to the surface (3).

We consider the coordinate transformation

$$x_i = \alpha_i + \alpha_{ij}y_j, \qquad |\alpha_{ij}| \neq 0, \qquad y_k = \beta_k + \beta_{ki}x_i, \qquad \beta_k = -\beta_{ki}\alpha_i, \tag{5}$$

where α_{ij} and β_{ki} are arbitrary real nondegenerate mutually inverse matrices: $\beta_{ki}\alpha_{ij} = \delta_{kj}$ and $\alpha_{ij}\beta_{jk} = \delta_{ik}$ [δ_{ik} is the Kronecker symbol (unit matrix)]. Formulas (5) determine the general affine transformation of coordinates. The

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constants α_i and β_k correspond to shifts along the axes, i.e., determine the coordinate origin. From Eqs. (5), we obtain

$$\frac{\partial}{\partial y_k} = \frac{\partial}{\partial x_i}\,\alpha_{ik}, \qquad \frac{\partial}{\partial x_s} = \frac{\partial}{\partial y_k}\,\beta_{ks}. \tag{6}$$

Taking into account Eqs. (6), from the first equation of (1) we obtain

$$\frac{\partial}{\partial y_k}\,\beta_{kj}\sigma_{ij} + F_i = 0, \qquad \frac{\partial}{\partial y_k}\,\beta_{li}\beta_{kj}\sigma_{ij} + \beta_{li}F_i = 0,$$

i.e.,

$$\frac{\partial}{\partial y_k}\,\tau_{lk} + \tilde{F}_l = 0, \tag{7}$$

where

$$\tau_{lk} = \beta_{li}\beta_{kj}\sigma_{ij}, \qquad \tilde{F}_l = \beta_{li}F_i, \qquad \sigma_{ij} = \alpha_{ip}\alpha_{jq}\tau_{pq}, \qquad F_k = \alpha_{kl}\tilde{F}_l. \tag{8}$$

Thus, the stresses and body forces are transformed by formulas (8). The form of the equations of equilibrium (7) is preserved in the new variables.

Let the strains be transformed by the formulas

$$\varepsilon_{ij} = \beta_{ri}\beta_{sj}e_{rs}, \qquad e_{pq} = \alpha_{ip}\alpha_{jq}\varepsilon_{ij}. \tag{9}$$

With allowance for Eqs. (8) and (9), we obtain the specific strain energy

$$2\Phi = \sigma_{ij}\varepsilon_{ij} = \alpha_{ip}\alpha_{jq}\tau_{pq}\beta_{ri}\beta_{sj}e_{rs} = \delta_{rp}\delta_{sq}\tau_{pq}e_{rs} = \tau_{rs}e_{rs}.$$

It is obvious that the strain energy 2Φ remains unchanged.

From Eqs. (1) and (9), we obtain

$$e_{pq} = \frac{1}{2}\,\alpha_{ip}\alpha_{jq}(\partial_i u_j + \partial_j u_i) = \frac{1}{2}\,(\partial_i\alpha_{ip}(\alpha_{jq}u_j) + \partial_j\alpha_{jq}(\alpha_{ip}u_i)) = \frac{1}{2}\Big(\frac{\partial}{\partial y_p}\,v_q + \frac{\partial}{\partial y_q}\,v_p\Big), \tag{10}$$

where

$$v_p = \alpha_{ip}u_i, \qquad u_k = \beta_{pk}v_p. \tag{11}$$

Thus, strains (10) written in the new variables are of the same form as in Eqs. (1).

Generalized Hooke's law (1) and Eqs. (8) and (9) imply that

$$\sigma_{ij} = A_{ijkl}\varepsilon_{kl} = A_{ijkl}\beta_{rk}\beta_{sl}e_{rs}, \qquad \beta_{pi}\beta_{qj}\sigma_{ij} = \beta_{pi}\beta_{qj}A_{ijkl}\beta_{rk}\beta_{sl}e_{rs},$$

i.e.,

$$\tau_{pq} = \tilde{A}_{pqrs}e_{rs}, \tag{12}$$

where

$$\tilde{A}_{pqrs} = \beta_{pi}\beta_{qj}A_{ijkl}\beta_{rk}\beta_{sl}, \qquad A_{ijkl} = \alpha_{ip}\alpha_{jq}\tilde{A}_{pqrs}\alpha_{kr}\alpha_{ls}. \tag{13}$$

One can see that the form of Hooke's law (12) is preserved and the elastic moduli are transformed by formulas (13).

The equation in displacements (2) can be combined with Eqs. (6), (8), and (11) to give

$$\beta_{ri}\beta_{sj}A_{i(kl)j}\beta_{pk}\beta_{ql}\tilde{\partial}_{pq}v_s + \beta_{ri}F_i = 0,$$

i.e.,

$$\tilde{A}_{r(pq)s}\tilde{\partial}_{pq}v_s + \tilde{F}_r = 0, \qquad \tilde{\partial}_{pq} = \frac{\partial^2}{\partial y_p \partial y_q}, \tag{14}$$

where

$$\tilde{A}_{r(pq)s} = \beta_{ri}\beta_{sj}A_{i(kl)j}\beta_{pk}\beta_{ql}, \qquad A_{i(kl)j} = \alpha_{ir}\alpha_{js}\tilde{A}_{r(pq)s}\alpha_{kp}\alpha_{lq}. \tag{15}$$

Thus, the equations in displacements (14) remain invariant and the coefficients are transformed by formulas (15).

The components of the unit normal to the boundary surface (3) are given by

$$n_i = f_i/\sqrt{f_k f_k}, \qquad f_i = \partial_i f.$$

After transformation, the equation of the boundary surface becomes

$$f(x_1, x_2, x_3) = f(\alpha_1 + \alpha_{1j} y_j,\ \alpha_2 + \alpha_{2j} y_j,\ \alpha_3 + \alpha_{3j} y_j) = \tilde{f}(y_1, y_2, y_3) = 0.$$

The components of the gradient vector $\tilde{f}_k = \tilde{\partial}_k \tilde{f} = (\partial_i f)\alpha_{ik} = f_i \alpha_{ik}$ and the unit normal are written as

$$\tilde{n}^*_k = \tilde{f}_k / \sqrt{\tilde{f}_l \tilde{f}_l} = f_i \alpha_{ik} / \sqrt{f_i \alpha_{il} f_s \alpha_{sl}}. \tag{16}$$

With allowance for Eqs. (8), the first boundary condition in (4) yields

$$\alpha_{ip}\alpha_{jq}\tau_{pq} n_j = p_i, \qquad \tau_{sq}\alpha_{jq} n_j = \beta_{si} p_i, \qquad \tau_{sq}\tilde{n}_q = \tilde{p}_s, \tag{17}$$

where

$$\tilde{n}_q = \alpha_{jq} n_j, \qquad n_k = \beta_{qk}\tilde{n}_q, \qquad \tilde{p}_s = \beta_{si} p_i, \qquad p_k = \alpha_{kl}\tilde{p}_l. \tag{18}$$

In Eq. (16), $\tilde{n}^*_k$ is the unit normal to the surface $\tilde{f}(y_s) = 0$; in Eqs. (17) and (18), $\tilde{n}_q = f_j \alpha_{jq} / \sqrt{f_k f_k} = n_j \alpha_{jq}$ is the nonunit normal: $\tilde{n}_q \tilde{n}_q = n_i n_j \alpha_{iq} \alpha_{jq} \neq 1$ if α_{iq} is a nonorthogonal matrix.

By virtue of Eqs. (11), from the second condition in (4), we obtain

$$\alpha_{ip} u_i = \alpha_{ip} u^0_i, \qquad v_p = v^0_p. \tag{19}$$

Hence, in the transformed region, the boundary conditions are of the form (17) or (19) similar to that of the starting conditions (4).

Thus, for arbitrary affine transformations of coordinates (5) with a nondegenerate matrix α_{ij} and corresponding induced transformations of stresses and body forces (8), strains (9) and (10), displacements (11), elastic moduli (13) and (15), and boundary conditions (18) and (19), Eqs. (1), (2), and (4) of the linear theory of elasticity remain invariant in the new variables [see (7), (10), (12), (14), (17), and (19)]. In the process, the specific strain energy 2Φ remains also unchanged.

The formulas given above can be used to model an anisotropic material by another material [3, 5, 7]. The solution of the boundary-value problem for one material being known, one can use the corresponding transformation formulas to obtain the solution of the boundary-value problem in the transformed region for the other material.

With allowance for symmetry of the elastic-modulus tensor, we write formulas (13) as

$$\begin{gathered} A_{ijkl} = \frac{1}{2}(\alpha_{ip}\alpha_{jq} + \alpha_{iq}\alpha_{jp})\tilde{A}_{pqrs}\frac{1}{2}(\alpha_{kr}\alpha_{ls} + \alpha_{ks}\alpha_{lr}) = \alpha_{ijpq}\tilde{A}_{pqrs}\alpha_{klrs}, \\ \tilde{A}_{pqrs} = \frac{1}{2}(\beta_{pi}\beta_{qj} + \beta_{pj}\beta_{qi})A_{ijkl}\frac{1}{2}(\beta_{rk}\beta_{sl} + \beta_{rl}\beta_{sk}) = \beta_{pqij}A_{ijkl}\beta_{rskl}, \\ \alpha_{ijpq} = \frac{1}{2}(\alpha_{ip}\alpha_{jq} + \alpha_{iq}\alpha_{jp}), \qquad \beta_{pqij} = \frac{1}{2}(\beta_{pi}\beta_{qj} + \beta_{pj}\beta_{qi}). \end{gathered} \tag{20}$$

Using formulas for passing from two subscripts to one subscript for tensors symmetric with respect to two indices

$$h_{11} = h_1, \qquad h_{22} = h_2, \qquad h_{33} = h_3,$$

$$\sqrt{2}h_{23} = \sqrt{2}h_{32} = h_4, \qquad \sqrt{2}h_{13} = \sqrt{2}h_{31} = h_5, \qquad \sqrt{2}h_{12} = \sqrt{2}h_{21} = h_6,$$

we write Eqs. (20) in the matrix form

$$A = \tilde{\alpha}\tilde{A}\tilde{\alpha}', \qquad \tilde{A} = \tilde{\beta}A\tilde{\beta}', \qquad A^{-1} = \tilde{\beta}'\tilde{A}^{-1}\tilde{\beta}, \qquad \tilde{A}^{-1} = \tilde{\alpha}' A^{-1}\tilde{\alpha}. \tag{21}$$

Here the prime denotes the transposition of the matrix. All matrices in Eqs. (21) are 6×6 matrices. The compliance matrix A^{-1} is inverse to the elastic-modulus matrix A. The matrices A and A^{-1} are symmetric and positively definite. The matrices corresponding to the tensors α_{ijpq} and β_{pqij} are denoted by $\tilde{\alpha}$ and $\tilde{\beta}$, respectively.

Formulas (21) are congruent transformations [13] of the matrices A, $\tilde{A}$, A^{-1}, and $\tilde{A}^{-1}$ determining the elastic properties of arbitrary anisotropic materials. Transformations (21) split all matrices (materials) A into classes of congruent (equivalent) matrices.

Likewise, relations (15) written in the matrix form

$$\tilde{A}^* = \tilde{\beta}A^*\tilde{\beta}', \qquad A^* = \tilde{\alpha}\tilde{A}^*\tilde{\alpha}',$$

are also congruent transformations of the matrices A^* and $\tilde{A}^*$ corresponding to the tensors of coefficients $A_{i(kl)j}$ and $\tilde{A}_{r(pq)s}$ in Eqs. (2) and (14).

The matrices $\tilde{\alpha}_{ij}$ (accordingly $\tilde{\beta}_{ij}$) have the form [14]

$$\tilde{\alpha}_{ij} = \begin{bmatrix} \alpha_{11}^2 & \alpha_{12}^2 & \alpha_{13}^2 & \sqrt{2}\alpha_{12}\alpha_{13} & \sqrt{2}\alpha_{11}\alpha_{13} & \sqrt{2}\alpha_{11}\alpha_{12} \\ \alpha_{21}^2 & \alpha_{22}^2 & \alpha_{23}^2 & \sqrt{2}\alpha_{22}\alpha_{23} & \sqrt{2}\alpha_{21}\alpha_{23} & \sqrt{2}\alpha_{21}\alpha_{22} \\ \alpha_{31}^2 & \alpha_{32}^2 & \alpha_{33}^2 & \sqrt{2}\alpha_{32}\alpha_{33} & \sqrt{2}\alpha_{31}\alpha_{33} & \sqrt{2}\alpha_{31}\alpha_{32} \\ \sqrt{2}\alpha_{21}\alpha_{31} & \sqrt{2}\alpha_{22}\alpha_{32} & \sqrt{2}\alpha_{23}\alpha_{33} & \alpha_{22}\alpha_{33}+\alpha_{23}\alpha_{32} & \alpha_{21}\alpha_{33}+\alpha_{23}\alpha_{31} & \alpha_{21}\alpha_{32}+\alpha_{22}\alpha_{31} \\ \sqrt{2}\alpha_{11}\alpha_{31} & \sqrt{2}\alpha_{12}\alpha_{32} & \sqrt{2}\alpha_{13}\alpha_{33} & \alpha_{12}\alpha_{33}+\alpha_{13}\alpha_{32} & \alpha_{11}\alpha_{33}+\alpha_{13}\alpha_{31} & \alpha_{11}\alpha_{32}+\alpha_{12}\alpha_{31} \\ \sqrt{2}\alpha_{11}\alpha_{21} & \sqrt{2}\alpha_{12}\alpha_{22} & \sqrt{2}\alpha_{13}\alpha_{23} & \alpha_{12}\alpha_{23}+\alpha_{13}\alpha_{22} & \alpha_{11}\alpha_{23}+\alpha_{13}\alpha_{21} & \alpha_{11}\alpha_{22}+\alpha_{12}\alpha_{21} \end{bmatrix}. \tag{22}$$

It is obvious that $\tilde{\alpha}$ and $\tilde{\beta}$ are mutually inverse matrices:

$$\tilde{\alpha}\tilde{\beta} = \alpha_{ijpq}\beta_{pqkl} = \frac{1}{2}\,(\alpha_{ip}\alpha_{jq} + \alpha_{iq}\alpha_{jp})\frac{1}{2}\,(\beta_{pk}\beta_{ql} + \beta_{pl}\beta_{qk}) = \frac{1}{2}\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) = \delta_{ijkl},$$

$$\tilde{\beta}\tilde{\alpha} = \beta_{pqij}\alpha_{ijrs} = \frac{1}{2}\,(\beta_{pi}\beta_{qj} + \beta_{pj}\beta_{qi})\frac{1}{2}\,(\alpha_{ir}\alpha_{js} + \alpha_{is}\alpha_{jr}) = \frac{1}{2}\,(\delta_{pr}\delta_{qs} + \delta_{ps}\delta_{qr}) = \delta_{pqrs}.$$

If α_{ij} are orthogonal matrices ($\alpha_{ij}\alpha_{ik} = \delta_{jk}$), the six-dimensional matrices $\tilde{\alpha}_{ij}$ (22) are orthogonal as well.

In the case of orthogonal matrices α_{ij}, formulas (21) imply that anisotropic materials are divided into seven syngonies, i.e., there exists a classification (invariance) with respect to subgroups of the orthogonal group of transformations.

For anisotropic materials, the eigenstates T are determined by relations of the form $A = T\Lambda T'$ (Λ is a diagonal matrix), i.e., by congruent transformations with the orthogonal matrix of eigenstates T. The theory of elasticity also involves a congruent transformation of the form $A = CDC'$ with a lower triangular matrix C and a diagonal matrix D. It is, therefore, of interest to classify anisotropic materials with respect to the group of affine, congruent transformations of the form (20), (21).

Let the matrix α_{ip} be the product of two matrices: $\alpha_{ip} = \alpha_{is}^{(2)}\alpha_{sp}^{(1)}$. In this case, it follows from the definition of α_{ijpq} (20) that

$$\alpha_{ijpq} = \frac{1}{2}\,(\alpha_{is}^{(2)}\alpha_{sp}^{(1)}\alpha_{jr}^{(2)}\alpha_{rq}^{(1)} + \alpha_{is}^{(2)}\alpha_{sq}^{(1)}\alpha_{jr}^{(2)}\alpha_{rp}^{(1)})$$

$$= \frac{1}{2}\,(\alpha_{ir}^{(2)}\alpha_{js}^{(2)} + \alpha_{is}^{(2)}\alpha_{jr}^{(2)})\frac{1}{2}\,(\alpha_{rp}^{(1)}\alpha_{sq}^{(1)} + \alpha_{rq}^{(1)}\alpha_{sp}^{(1)}) = \alpha_{ijrs}^{(2)}\alpha_{rspq}^{(1)}.$$

Thus, the group of nondegenerate affine transformations $\alpha_{ip} = \alpha_{is}^{(2)}\alpha_{sp}^{(1)}$ corresponds to the linear representation of the group $\alpha_{ijpq} = \alpha_{ijrs}^{(2)}\alpha_{rspq}^{(1)}$.

Godunov [15] showed that any nondegenerate transformation α with a positive determinant is represented in the form $\alpha = UKV$, where U and V are the rotation matrices: $UU' = E$ and $VV' = E$ (E is the unit matrix), $|U| = |V| = 1$, and K is the diagonal matrix: $K = \mathrm{diag}\,(k_1, k_2, k_3)$, $k_i > 0$. A similar representation is valid for the inverse matrix $\beta = \alpha^{-1} = V'K^{-1}U'$.

Thus, the transformation β can be written as a product of three transformations

$$\beta = \beta^{(3)}\beta^{(2)}\beta^{(1)}, \tag{23}$$

where $\beta^{(1)}$ and $\beta^{(3)}$ are the rotation matrices and $\beta^{(2)}$ is the diagonal matrix:

$$\beta^{(2)} = \mathrm{diag}\,(\beta_1, \beta_2, \beta_3), \qquad \beta_i > 0. \tag{24}$$

The matrix $\tilde{\beta}_{ij}$ is of the form of (22), where α_{ij} should be replaced by β_{ij}. In the case of (23), the representation of $\tilde{\beta}$ becomes

$$\tilde{\beta} = \tilde{\beta}^{(3)}\tilde{\beta}^{(2)}\tilde{\beta}^{(1)} \tag{25}$$

and Eqs. (21) and (25) yield

$$\tilde{A} = \tilde{\beta}^{(3)}\tilde{\beta}^{(2)}\tilde{\beta}^{(1)}A\tilde{\beta}^{(1)\prime}\tilde{\beta}^{(2)\prime}\tilde{\beta}^{(3)\prime}. \tag{26}$$

We now write the matrix $\tilde{\beta}^{(2)}$ with allowance for Eq. (24):

$$\tilde{\beta}^{(2)} = \mathrm{diag}\,(\beta_1^2,\, \beta_2^2,\, \beta_3^2,\, \beta_2\beta_3,\, \beta_1\beta_3,\, \beta_1\beta_2). \tag{27}$$

It follows from Eq. (26) that the general congruent transformation consists of the orthogonal transformation $\tilde{\beta}^{(1)}$, extension along the axes with matrix (27), and one more orthogonal transformation $\tilde{\beta}^{(3)}$ performed in succession.

The determinant of matrix (23) is written as $|\beta| = |\beta^{(3)}||\beta^{(2)}||\beta^{(1)}| = |\beta^{(2)}| = \beta_1\beta_2\beta_3$ and that of matrix (25) is $|\tilde{\beta}| = |\tilde{\beta}^{(3)}||\tilde{\beta}^{(2)}||\tilde{\beta}^{(1)}| = |\tilde{\beta}^{(2)}| = (\beta_1\beta_2\beta_3)^4$. The orthogonal transformation $\tilde{\beta}^{(1)}$ (26) transforms the starting matrix A from one orthogonal coordinate system to another: $\tilde{A}^{(1)} = \tilde{\beta}^{(1)} A \tilde{\beta}^{(1)\prime}$. Therefore, $\tilde{A}^{(1)}$ can be used as the starting matrix A. Using Eq. (27), we obtain

$$\tilde{A}^{(2)} = \tilde{\beta}^{(2)} A \tilde{\beta}^{(2)\prime} = \begin{bmatrix} \beta_1^4 A_{11} & & & & & \\ \beta_1^2\beta_2^2 A_{21} & \beta_2^4 A_{22} & & & \text{sym} & \\ \beta_1^2\beta_3^2 A_{31} & \beta_2^2\beta_3^2 A_{32} & \beta_3^4 A_{33} & & & \\ \beta_1^2\beta_2\beta_3 A_{41} & \beta_2^3\beta_3 A_{42} & \beta_2\beta_3^3 A_{43} & \beta_2^2\beta_3^2 A_{44} & & \\ \beta_1^3\beta_3 A_{51} & \beta_1\beta_2^2\beta_3 A_{52} & \beta_1\beta_3^3 A_{53} & \beta_1\beta_2\beta_3^2 A_{54} & \beta_1^2\beta_3^2 A_{55} & \\ \beta_1^3\beta_2 A_{61} & \beta_1\beta_2^3 A_{62} & \beta_1\beta_2\beta_3^2 A_{63} & \beta_1\beta_2^2\beta_3 A_{64} & \beta_1^2\beta_2\beta_3 A_{65} & \beta_1^2\beta_2^2 A_{66} \end{bmatrix}. \tag{28}$$

At the third step, the orthogonal transformation $\tilde{\beta}^{(3)}$ transforms the matrix $\tilde{A}^{(2)}$ (28) into a certain new orthogonal coordinate system: $\tilde{A}^{(3)} = \tilde{\beta}^{(3)} \tilde{A}^{(2)} \tilde{\beta}^{(3)\prime}$.

In the dynamic case, the equations in displacements (2) become

$$(A_{i(kl)j}\partial_{kl} - \rho c^2 \delta_{ij}\partial_{44}) u_j + F_i = 0, \tag{29}$$

where $\partial_4 = \partial/\partial x_4$, $x_4 = ct$ (c is a certain constant having the dimension of velocity and t is the time), and ρ is the constant density of the material. For transformations (5), from Eq. (29) we obtain the following equation instead of Eq. (14):

$$(\tilde{A}_{r(pq)s}\tilde{\partial}_{pq} - \rho c^2 \beta_{ri}\beta_{si}\partial_{44}) v_s + \tilde{F}_r = 0. \tag{30}$$

In (30), the expression $\beta_{ri}\beta_{si}$ is given by $\beta_{ri}\beta_{si} = \beta\beta'$; if $\beta = \beta^{(3)}\beta^{(2)}\beta^{(1)}$ [see (23)], then

$$\beta\beta' = \beta^{(3)}\beta^{(2)}\beta^{(1)}\beta^{(1)\prime}\beta^{(2)\prime}\beta^{(3)\prime} = \beta^{(3)}\beta^{(2)}\beta^{(2)\prime}\beta^{(3)\prime}.$$

With allowance for Eq. (24), we obtain $\beta^{(2)}\beta^{(2)\prime} = \mathrm{diag}\,(\beta_1^2,\, \beta_2^2,\, \beta_3^2)$, and hence,

$$\beta\beta' = \beta^{(3)}\mathrm{diag}\,(\beta_1^2,\, \beta_2^2,\, \beta_3^2)\beta^{(3)\prime} = \beta_1^2\beta_{r1}^{(3)}\beta_{s1}^{(3)} + \beta_2^2\beta_{r2}^{(3)}\beta_{s2}^{(3)} + \beta_3^2\beta_{r3}^{(3)}\beta_{s3}^{(3)}.$$

The last expression has the form of a spherical tensor provided that $\beta_1^2 = \beta_2^2 = \beta_3^2$, then $\beta_{ri}\beta_{si} = \beta_1^2\beta_{ri}^{(3)}\beta_{si}^{(3)} = \beta_1^2\delta_{rs}$. This implies that Eq. (30) preserves its form if the transformation β_{ri} is orthogonal up to the coefficient β_1. Thus, in the dynamic case, the transformation that preserves the equations should be orthogonal rather than affine.

For orthogonal transformations of the coordinate system, invariants of the elastic-modulus tensor A_{ijkl} and infinitesimal operators were considered in [14, 16]. We determine infinitesimal operators for transformations (27) and (28). For a differential infinitesimal transformation, the matrix $\beta^{(2)}$ in (24) differs from the unit matrix by an infinitely small quantity, i.e., in differentiating with respect to β_i, one should set $\beta_i = 1$. From Eqs. (28), we obtain

$$\begin{aligned}
&dA_{11} = 4A_{11}d\beta_1, && \\
&dA_{21} = 2A_{21}(d\beta_1 + d\beta_2), && dA_{22} = 4A_{22}d\beta_2, \\
&dA_{31} = 2A_{31}(d\beta_1 + d\beta_3), && dA_{32} = 2A_{32}(d\beta_2 + d\beta_3), \\
&dA_{41} = A_{41}(2d\beta_1 + d\beta_2 + d\beta_3), && dA_{42} = A_{42}(3d\beta_2 + d\beta_3), \\
&dA_{51} = A_{51}(3d\beta_1 + d\beta_3), && dA_{52} = A_{52}(d\beta_1 + 2d\beta_2 + d\beta_3), \\
&dA_{61} = A_{61}(3d\beta_1 + d\beta_2); && dA_{62} = A_{62}(d\beta_1 + 3d\beta_2); \\
&dA_{33} = 4A_{33}d\beta_3, && \\
&dA_{43} = A_{43}(d\beta_2 + 3d\beta_3), && dA_{44} = 2A_{44}(d\beta_2 + d\beta_3), \\
&dA_{53} = A_{53}(d\beta_1 + 3d\beta_3), && dA_{54} = A_{54}(d\beta_1 + d\beta_2 + 2d\beta_3), \\
&dA_{63} = A_{63}(d\beta_1 + d\beta_2 + 2d\beta_3); && dA_{64} = A_{64}(d\beta_1 + 2d\beta_2 + d\beta_3); \\
&dA_{55} = 2A_{55}(d\beta_1 + d\beta_3), && \\
&dA_{65} = A_{65}(2d\beta_1 + d\beta_2 + d\beta_3); && dA_{66} = 2A_{66}(d\beta_1 + d\beta_2).
\end{aligned} \tag{31}$$

For Eqs. (31), we write the infinitesimal operators:

$$
\begin{aligned}
D_1 = {} & 4A_{11}\partial_{A_{11}} + 2A_{21}\partial_{A_{21}} + 2A_{31}\partial_{A_{31}} + 2A_{41}\partial_{A_{41}} + 3A_{51}\partial_{A_{51}} \\
& + 3A_{61}\partial_{A_{61}} + A_{52}\partial_{A_{52}} + A_{62}\partial_{A_{62}} + A_{53}\partial_{A_{53}} + A_{63}\partial_{A_{63}} \\
& + A_{54}\partial_{A_{54}} + A_{64}\partial_{A_{64}} + 2A_{55}\partial_{A_{55}} + 2A_{65}\partial_{A_{65}} + 2A_{66}\partial_{A_{66}}, \\
D_2 = {} & 2A_{21}\partial_{A_{21}} + A_{41}\partial_{A_{41}} + A_{61}\partial_{A_{61}} + 4A_{22}\partial_{A_{22}} + 2A_{32}\partial_{A_{32}} \\
& + 3A_{42}\partial_{A_{42}} + 2A_{52}\partial_{A_{52}} + 3A_{62}\partial_{A_{62}} + A_{43}\partial_{A_{43}} + A_{63}\partial_{A_{63}} \\
& + 2A_{44}\partial_{A_{44}} + A_{54}\partial_{A_{54}} + 2A_{64}\partial_{A_{64}} + A_{65}\partial_{A_{65}} + 2A_{66}\partial_{A_{66}}, \\
D_3 = {} & 2A_{31}\partial_{A_{31}} + A_{41}\partial_{A_{41}} + A_{51}\partial_{A_{51}} + 2A_{32}\partial_{A_{32}} + A_{42}\partial_{A_{42}} \\
& + A_{52}\partial_{A_{52}} + 4A_{33}\partial_{A_{33}} + 3A_{43}\partial_{A_{43}} + 3A_{53}\partial_{A_{53}} + 2A_{63}\partial_{A_{63}} \\
& + 2A_{44}\partial_{A_{44}} + 2A_{54}\partial_{A_{54}} + A_{64}\partial_{A_{64}} + 2A_{55}\partial_{A_{55}} + A_{65}\partial_{A_{65}}.
\end{aligned}
$$

The invariants of the tensor A_{ijkl} (matrix A_{ij}) are algebraically independent solutions of the closed system of differential equations

$$
D_1 f = 0, \qquad D_2 f = 0, \qquad D_3 f = 0. \tag{32}
$$

Since the infinitesimal operators form a Lie algebra, system (32) is closed. The closed system (32) of three differential equations with 21 variables A_{ij} has 18 algebraically independent solutions.

We write the general congruency conditions for two anisotropic materials whose elastic moduli are related by transformation (28). From (28), we obtain

$$
\beta_1 = \sqrt[4]{\frac{\tilde{A}_{11}}{A_{11}}}, \qquad \beta_2 = \sqrt[4]{\frac{\tilde{A}_{22}}{A_{22}}}, \qquad \beta_3 = \sqrt[4]{\frac{\tilde{A}_{33}}{A_{33}}} \tag{33}
$$

[the subscript (2) is omitted]. Elimination of the parameters β_i yields

$$
\begin{aligned}
&\frac{A_{21}}{\sqrt{A_{11}A_{22}}} = \frac{\tilde{A}_{21}}{\sqrt{\tilde{A}_{11}\tilde{A}_{22}}} = k_{21}, \\
&\frac{A_{31}}{\sqrt{A_{11}A_{33}}} = \frac{\tilde{A}_{31}}{\sqrt{\tilde{A}_{11}\tilde{A}_{33}}} = k_{31}, \qquad \frac{A_{32}}{\sqrt{A_{22}A_{33}}} = \frac{\tilde{A}_{32}}{\sqrt{\tilde{A}_{22}\tilde{A}_{33}}} = k_{32}, \\
&\frac{A_{41}}{\sqrt{A_{11}}\sqrt[4]{A_{22}A_{33}}} = \frac{\tilde{A}_{41}}{\sqrt{\tilde{A}_{11}}\sqrt[4]{\tilde{A}_{22}\tilde{A}_{33}}} = k_{41}, \qquad \frac{A_{42}}{\sqrt{A_{22}}\sqrt[4]{A_{22}A_{33}}} = \frac{\tilde{A}_{42}}{\sqrt{\tilde{A}_{22}}\sqrt[4]{\tilde{A}_{22}\tilde{A}_{33}}} = k_{42}, \\
&\frac{A_{51}}{\sqrt{A_{11}}\sqrt[4]{A_{11}A_{33}}} = \frac{\tilde{A}_{51}}{\sqrt{\tilde{A}_{11}}\sqrt[4]{\tilde{A}_{11}\tilde{A}_{33}}} = k_{51}, \qquad \frac{A_{52}}{\sqrt{A_{22}}\sqrt[4]{A_{11}A_{33}}} = \frac{\tilde{A}_{52}}{\sqrt{\tilde{A}_{22}}\sqrt[4]{\tilde{A}_{11}\tilde{A}_{33}}} = k_{52}, \\
&\frac{A_{61}}{\sqrt{A_{11}}\sqrt[4]{A_{11}A_{22}}} = \frac{\tilde{A}_{61}}{\sqrt{\tilde{A}_{11}}\sqrt[4]{\tilde{A}_{11}\tilde{A}_{22}}} = k_{61}; \qquad \frac{A_{62}}{\sqrt{A_{22}}\sqrt[4]{A_{11}A_{22}}} = \frac{\tilde{A}_{62}}{\sqrt{\tilde{A}_{22}}\sqrt[4]{\tilde{A}_{11}\tilde{A}_{22}}} = k_{62}; \\
&\frac{A_{43}}{\sqrt{A_{33}}\sqrt[4]{A_{22}A_{33}}} = \frac{\tilde{A}_{43}}{\sqrt{\tilde{A}_{33}}\sqrt[4]{\tilde{A}_{22}\tilde{A}_{33}}} = k_{43}, \qquad \frac{A_{44}}{\sqrt{A_{22}A_{33}}} = \frac{\tilde{A}_{44}}{\sqrt{\tilde{A}_{22}\tilde{A}_{33}}} = k_{44}, \\
&\frac{A_{53}}{\sqrt{A_{33}}\sqrt[4]{A_{11}A_{33}}} = \frac{\tilde{A}_{53}}{\sqrt{\tilde{A}_{33}}\sqrt[4]{\tilde{A}_{11}\tilde{A}_{33}}} = k_{53}, \qquad \frac{A_{54}}{\sqrt{A_{33}}\sqrt[4]{A_{11}A_{22}}} = \frac{\tilde{A}_{54}}{\sqrt{\tilde{A}_{33}}\sqrt[4]{\tilde{A}_{11}\tilde{A}_{22}}} = k_{54}, \\
&\frac{A_{63}}{\sqrt{A_{33}}\sqrt[4]{A_{11}A_{22}}} = \frac{\tilde{A}_{63}}{\sqrt{\tilde{A}_{33}}\sqrt[4]{\tilde{A}_{11}\tilde{A}_{22}}} = k_{63}; \qquad \frac{A_{64}}{\sqrt{A_{22}}\sqrt[4]{A_{11}A_{33}}} = \frac{\tilde{A}_{64}}{\sqrt{\tilde{A}_{22}}\sqrt[4]{\tilde{A}_{11}\tilde{A}_{33}}} = k_{64}; \\
&\frac{A_{55}}{\sqrt{A_{11}A_{33}}} = \frac{\tilde{A}_{55}}{\sqrt{\tilde{A}_{11}\tilde{A}_{33}}} = k_{55}, \\
&\frac{A_{65}}{\sqrt{A_{11}}\sqrt[4]{A_{22}A_{33}}} = \frac{\tilde{A}_{65}}{\sqrt{\tilde{A}_{11}}\sqrt[4]{\tilde{A}_{22}\tilde{A}_{33}}} = k_{65}; \qquad \frac{A_{66}}{\sqrt{A_{11}A_{22}}} = \frac{\tilde{A}_{66}}{\sqrt{\tilde{A}_{11}\tilde{A}_{22}}} = k_{66}.
\end{aligned} \tag{34}
$$

One can verify that 18 quantities k_{ij} (34) satisfy Eqs. (32), i.e., are invariants of transformation (28). Thus, two anisotropic materials are congruent if the elastic moduli A_{ij} and $\tilde{A}_{ij}$ are related by Eqs. (34), the parameters of the transformation β_i being determined by formulas (33).

For an isotropic material, the tensor A_{ijkl} is given by

$$A_{ijkl} = \lambda\delta_{ij}\delta_{kl} + 2\mu\delta_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{35}$$

where λ and μ are the Lamé constants. Using Eqs. (13) and (35), we obtain the elastic-modulus tensor of anisotropic materials congruent to the isotropic material:

$$\begin{aligned} \tilde{A}_{pqrs} &= \beta_{pi}\beta_{qj}[\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]\beta_{rk}\beta_{sl} \\ &= \lambda\beta_{pi}\beta_{qi}\beta_{rk}\beta_{sk} + \mu(\beta_{pk}\beta_{rk}\beta_{ql}\beta_{sl} + \beta_{pl}\beta_{sl}\beta_{qk}\beta_{rk}) = \lambda b_{pq}b_{rs} + \mu(b_{pr}b_{qs} + b_{ps}b_{qr}) \end{aligned} \tag{36}$$

($b_{pq} = \beta_{pi}\beta_{qi}$). Thus, the elastic moduli (36) of the congruent anisotropic material have the same form as moduli (35) for an isotropic material, but Eq. (36) contains an arbitrary symmetric nondegenerate tensor of the form $b_{pq} = \beta_{pi}\beta_{qi}$ instead of the unit tensor δ_{ij}.

Let us show that quantities (36) written in an arbitrary coordinate system determine a special orthotropic material. The general congruent transformation is given by Eq. (26). The orthogonal transformation $\tilde{\beta}^{(1)}$ does not change the moduli of the isotropic material (35). Using Eqs. (28), we obtain the following relation for the isotropic material:

$$\tilde{A}^{(2)}_{pq} = \begin{bmatrix} \beta_1^4 A_{11} & & & & & \\ \beta_1^2\beta_2^2 A_{21} & \beta_2^4 A_{11} & & & \text{sym} & \\ \beta_1^2\beta_3^2 A_{21} & \beta_2^2\beta_3^2 A_{21} & \beta_3^4 A_{11} & & & \\ 0 & 0 & 0 & \beta_2^2\beta_3^2(A_{11} - A_{21}) & & \\ 0 & 0 & 0 & 0 & \beta_1^2\beta_3^2(A_{11} - A_{21}) & \\ 0 & 0 & 0 & 0 & 0 & \beta_1^2\beta_2^2(A_{11} - A_{21}) \end{bmatrix}$$

$$= \begin{bmatrix} \beta_1^4(\lambda + 2\mu) & & & & & \\ \beta_1^2\beta_2^2\lambda & \beta_2^4(\lambda + 2\mu) & & & \text{sym} & \\ \beta_1^2\beta_3^2\lambda & \beta_2^2\beta_3^2\lambda & \beta_3^4(\lambda + 2\mu) & & & \\ 0 & 0 & 0 & \beta_2^2\beta_3^2 2\mu & & \\ 0 & 0 & 0 & 0 & \beta_1^2\beta_3^2 2\mu & \\ 0 & 0 & 0 & 0 & 0 & \beta_1^2\beta_2^2 2\mu \end{bmatrix}. \tag{37}$$

Expression (37) contains five parameters: β_1, β_2, β_3, λ, and μ; matrix (37) corresponds to an orthotropic material of special form. Transformations with the orthogonal matrix $\tilde{\beta}^{(3)}$ (26) transform Eqs. (37) into the general coordinate system (36): $\tilde{A}^{(3)} = \tilde{\beta}^{(3)}\tilde{A}^{(2)}\tilde{\beta}^{(3)\prime}$. In the process, three more parameters determining the orthogonal coordinate system are added.

The isotropic-material conditions (34) imply that

$$\begin{gathered} \frac{\lambda}{\lambda + 2\mu} = \frac{A_{21}}{A_{11}} = \frac{\tilde{A}_{21}}{\sqrt{\tilde{A}_{11}\tilde{A}_{22}}} = \frac{\tilde{A}_{31}}{\sqrt{\tilde{A}_{11}\tilde{A}_{33}}} = \frac{\tilde{A}_{32}}{\sqrt{\tilde{A}_{22}\tilde{A}_{33}}} = k_{21}, \\ \frac{2\mu}{\lambda + 2\mu} = \frac{A_{11} - A_{21}}{A_{11}} = \frac{\tilde{A}_{44}}{\sqrt{\tilde{A}_{22}\tilde{A}_{33}}} = \frac{\tilde{A}_{55}}{\sqrt{\tilde{A}_{11}\tilde{A}_{33}}} = \frac{\tilde{A}_{66}}{\sqrt{\tilde{A}_{11}\tilde{A}_{22}}} = k_{44} = 1 - k_{21}. \end{gathered} \tag{38}$$

The positive-definiteness conditions $3\lambda + 2\mu > 0$ and $2\mu > 0$ impose restrictions on the parameter k_{21}: $-1/2 < k_{21} < 1$. Now, taking into account Eqs. (38), we write matrix (37) for an orthotropic material congruent to the isotropic material as

$$\tilde{A}_{pq} = \begin{bmatrix} \tilde{A}_{11} & & & & & \\ k_{21}\sqrt{\tilde{A}_{11}\tilde{A}_{22}} & \tilde{A}_{22} & & & \text{sym} & \\ k_{21}\sqrt{\tilde{A}_{11}\tilde{A}_{33}} & k_{21}\sqrt{\tilde{A}_{22}\tilde{A}_{33}} & \tilde{A}_{33} & & & \\ 0 & 0 & 0 & k_{44}\sqrt{\tilde{A}_{22}\tilde{A}_{33}} & & \\ 0 & 0 & 0 & 0 & k_{44}\sqrt{\tilde{A}_{11}\tilde{A}_{33}} & \\ 0 & 0 & 0 & 0 & 0 & k_{44}\sqrt{\tilde{A}_{11}\tilde{A}_{22}} \end{bmatrix}. \tag{39}$$

Expression (39) contains four free parameters: $\tilde{A}_{11}$, $\tilde{A}_{22}$, $\tilde{A}_{33}$, and k_{21}.

If $\tilde{A}_{11} = \tilde{A}_{22}$ in Eq. (39), for a transversely-isotropic material congruent to the isotropic material, we obtain

$$\tilde{A}_{pq} = \begin{bmatrix} \tilde{A}_{11} & & & & & \\ k_{21}\tilde{A}_{11} & \tilde{A}_{11} & & & \text{sym} & \\ k_{21}\sqrt{\tilde{A}_{11}\tilde{A}_{33}} & k_{21}\sqrt{\tilde{A}_{11}\tilde{A}_{33}} & \tilde{A}_{33} & & & \\ 0 & 0 & 0 & k_{44}\sqrt{\tilde{A}_{11}\tilde{A}_{33}} & & \\ 0 & 0 & 0 & 0 & k_{44}\sqrt{\tilde{A}_{11}\tilde{A}_{33}} & \\ 0 & 0 & 0 & 0 & 0 & (1-k_{21})\tilde{A}_{11} \end{bmatrix}.$$

This expression contains three free parameters: $\tilde{A}_{11}$, $\tilde{A}_{33}$, and k_{21}.

We now find the matrix of the moduli of materials congruent to a transversely isotropic material. We assume that the orthogonal transformation $\tilde{\beta}^{(1)}$ (26) reduces the matrix of the moduli of the transversely isotropic material to the form (x_3 is the rotation axis)

$$A_{ij} = \begin{bmatrix} A_{11} & & & & & \\ A_{21} & A_{11} & & & \text{sym} & \\ A_{31} & A_{31} & A_{33} & & & \\ 0 & 0 & 0 & A_{44} & & \\ 0 & 0 & 0 & 0 & A_{44} & \\ 0 & 0 & 0 & 0 & 0 & A_{11}-A_{21} \end{bmatrix}. \tag{40}$$

Conditions (34) imply that

$$\frac{A_{21}}{A_{11}} = \frac{\tilde{A}_{21}}{\sqrt{\tilde{A}_{11}\tilde{A}_{22}}} = k_{21}, \qquad \frac{A_{31}}{\sqrt{A_{11}A_{33}}} = \frac{\tilde{A}_{31}}{\sqrt{\tilde{A}_{11}\tilde{A}_{33}}} = \frac{\tilde{A}_{32}}{\sqrt{\tilde{A}_{22}\tilde{A}_{33}}} = k_{31},$$
$$\frac{A_{44}}{\sqrt{A_{11}A_{33}}} = \frac{\tilde{A}_{44}}{\sqrt{\tilde{A}_{22}\tilde{A}_{33}}} = \frac{\tilde{A}_{55}}{\sqrt{\tilde{A}_{11}\tilde{A}_{33}}} = k_{44}, \qquad \frac{A_{11}-A_{21}}{A_{11}} = \frac{\tilde{A}_{66}}{\sqrt{\tilde{A}_{11}\tilde{A}_{22}}} = k_{66} = 1-k_{21}. \tag{41}$$

Thus, with allowance for Eqs. (41), the matrix of the moduli of the transversely isotropic material (40) is written as

$$A_{ij} = \begin{bmatrix} A_{11} & & & & & \\ k_{21}A_{11} & A_{11} & & & \text{sym} & \\ k_{31}\sqrt{A_{11}A_{33}} & k_{31}\sqrt{A_{11}A_{33}} & A_{33} & & & \\ 0 & 0 & 0 & k_{44}\sqrt{A_{11}A_{33}} & & \\ 0 & 0 & 0 & 0 & k_{44}\sqrt{A_{11}A_{33}} & \\ 0 & 0 & 0 & 0 & 0 & (1-k_{21})A_{11} \end{bmatrix}. \tag{42}$$

This material is congruent to an orthotropic material with the matrix of moduli

$$\tilde{A}_{pq} = \begin{bmatrix} \tilde{A}_{11} & & & & & \\ k_{21}\sqrt{\tilde{A}_{11}\tilde{A}_{22}} & \tilde{A}_{22} & & & \text{sym} & \\ k_{31}\sqrt{\tilde{A}_{11}\tilde{A}_{33}} & k_{31}\sqrt{\tilde{A}_{22}\tilde{A}_{33}} & \tilde{A}_{33} & & & \\ 0 & 0 & 0 & k_{44}\sqrt{\tilde{A}_{22}\tilde{A}_{33}} & & \\ 0 & 0 & 0 & 0 & k_{44}\sqrt{\tilde{A}_{11}\tilde{A}_{33}} & \\ 0 & 0 & 0 & 0 & 0 & (1-k_{21})\sqrt{\tilde{A}_{11}\tilde{A}_{22}} \end{bmatrix}. \tag{43}$$

Formula (42) contains five free parameters: A_{11}, A_{33}, k_{21}, k_{31}, k_{44}, whereas Eq. (43) contains six parameters: $\tilde{A}_{11}$, $\tilde{A}_{22}$, $\tilde{A}_{33}$, k_{21}, k_{31}, and k_{44}. The transformation coefficients are determined by formulas (33). If $\tilde{A}_{11} = \tilde{A}_{22}$ in Eq. (43), then Eqs. (42) and (43) correspond to two congruent transversely isotropic materials.

For a transversely isotropic material, the positive-definiteness conditions

$$A_{11} + A_{21} > 2A_{31}^2, \qquad A_{11} > A_{21}, \qquad A_{44} > 0$$

yield the following restrictions on the parameters k_{21}, k_{31}, and k_{44}:

$$2k_{31}^2 - 1 < k_{21} < 1, \qquad -1 < k_{31} < 1, \qquad k_{44} > 0.$$

It follows from Eqs. (28) and (34) that an orthotropic material is transformed into an orthotropic material:

$$\tilde{A}_{pq} = \begin{bmatrix} \beta_1^4 A_{11} & & & & & \\ \beta_1^2\beta_2^2 A_{21} & \beta_2^4 A_{22} & & & \text{sym} & \\ \beta_1^2\beta_3^2 A_{31} & \beta_2^2\beta_3^2 A_{32} & \beta_3^4 A_{33} & & & \\ 0 & 0 & 0 & \beta_2^2\beta_3^2 A_{44} & & \\ 0 & 0 & 0 & 0 & \beta_1^2\beta_3^2 A_{55} & \\ 0 & 0 & 0 & 0 & 0 & \beta_1^2\beta_2^2 A_{66} \end{bmatrix}.$$

Using formulas (28) and (34), one obtains the elastic moduli of congruent materials for the remaining syngonies [17]: cubic, trigonal, tetragonal, and monoclinic.

In summary, the relations obtained in this paper allow one to find solutions of boundary-value problems of the theory of elasticity not only for a single specific material but also for classes of congruent anisotropic or isotropic materials. Since the static equations of the linear theory of elasticity are invariant with respect to a group of affine transformations, one can use methods of continuous Lie groups [18] to obtain invariant solutions.

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